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SPATIAL ANALYTICITY OF SOLUTIONS OF A NONLOCAL PERTURBATION OF THE KDV EQUATION

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Abstract. Let $\mathcal{H}$ denote the Hilbert transform and $\eta \geq 0$. We show that if the initial data of the following problems $u_t + uu_x + u_{xxx} + \eta(\mathcal{H}u_x + \mathcal{H}u_{xxx}) = 0$, $u(\cdot, 0) = \phi(\cdot)$ and $v_t + \frac{1}{2}(v_x)^2 + v_{xxx} + \eta(\mathcal{H}v_x + \mathcal{H}v_{xxx}) = 0$, $v(\cdot, 0) = \psi(\cdot)$ has an analytic continuation to a strip containing the real axis, then the solution has the same property, although the width of the strip might diminish with time. When $\eta > 0$ and the initial data is complex-valued we prove local well-posedness of the two problems above in spaces of analytic functions, which implies the constancy over time of the radius of the strip of analyticity in the complex plane around the real axis.

1. Introduction

We are interested in studying spatial analyticity of solutions of the following problems:

$$u_t + uu_x + u_{xxx} + \eta(\mathcal{H}u_x + \mathcal{H}u_{xxx}) = 0,\ u(\cdot, 0) = \phi(\cdot), \tag{1}$$

$$v_t + \frac{1}{2}(v_x)^2 + v_{xxx} + \eta(\mathcal{H}v_x + \mathcal{H}v_{xxx}) = 0,\ v(\cdot, 0) = \psi(\cdot), \tag{2}$$

where $\mathcal{H}$ denotes the Hilbert transform given by $\mathcal{H}f(x) = \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{f(y)}{y-x}dy$ for $f \in \mathcal{S}(\mathbb{R})$ the Schwartz space of rapidly decreasing $C^\infty(\mathbb{R})$ functions, $\mathcal{P}$ represents the principal value of the integral and the parameter η is an arbitrary nonnegative number. It is known that $\widehat{(\mathcal{H}f)}(\xi) = i\mathrm{sgn}(\xi)\hat{f}(\xi)$, for all $f \in H^s(\mathbb{R})$, where

$$\mathrm{sgn}\,(\xi) = \begin{cases} -1, & \xi < 0, \\ 1, & \xi > 0. \end{cases}$$

Equation (1) was derived by Ostrovsky *et.al.* (see [9] for more details) to describe the radiational instability of long non-linear waves in a stratified fluid caused by internal wave radiation from a shear layer; the fourth term corresponds to the *wave amplification* and the fifth term represents *damping*. It models the motion of a homogeneous finite-thickness fluid layer with density δ_1, which moves at a constant speed U, slipping over an immobile infinitely deep stratified fluid with a density $\delta_2 > \delta_1$. The upper boundary of the layer is supposed to be rigid and the lower one is contiguous to the infinitely deep fluid. Here $u(x, t)$ is the deviation of the

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interface from its equilibrium position. Let us remark that some numerical results for periodic and solitary-wave solutions of equation (1) were obtained by Bao-Feng Feng and T. Kawahara [4].

The Cauchy problems associated to (1) and (2) were studied in [1], where it was proved that problems (1) and (2) are globally well-posed in $H^s(\mathbb{R})$ for $s \geq 1$, considering real-valued solutions.

In this paper we are interested in proving that if the initial condition of the problem (1) (resp. (2)) is analytic and has an analytic continuation to a strip containing the real axis, then the solution of (1) (resp. (2)) has the same property.

Section 2 is devoted to studying the case when the solutions are real-valued on the real axis at any time, and $\eta \geq 0$. Hence, the results obtained in [1] about the initial value problems associated to (1) and (2) will be helpful. We use the method developed by Kato&Masuda [8] which estimates certain families of Liapunov functions for the solutions, to prove global spatial analyticity of the solutions, but the width of the strip might decrease with time.

Section 3 shows that problems (1) and (2) admit a Gevrey-class analysis. For $\eta > 0$, we prove local well-posedness of problem (1) (resp. (2)) in $X^{\sigma,s}$ for $\sigma > 0$ and $s > 1/2$ (resp. $s > 3/2$); here, the initial data can be complex-valued. So, if the initial data of problem (1) (resp. (2)) is analytic and has an analytic continuation to a strip containing the real axis, then the solution of (1) (resp. (2)) has the same property, maintaining the width of the strip in time. It should be mentioned that it was recently proved by Grujić&Kalisch [5] a result on local well-posedness of the generalized KdV equation (KdV is an abbreviation for Korteweg-de Vries) in spaces of analytic functions on a strip containing the real axis without shrinking the width of the strip in time; their proof uses space-time estimates and Bourgain-type spaces. Here we do not make use of Bourgain spaces, we mainly use some properties of the Semigroup associated to the linear part of problem (1), namely Lemmas 3.1 and 3.2, to prove local well-posedness of problem (1) in $X^{\sigma,s}$. Moreover, proceeding as in [2], where Bona&Grujić studied some KdV-type equations, we prove for real-valued solutions and $\eta \geq 0$ that if the initial state belongs to a Gevrey class, then the solution of (1) (resp. (2)) remains in this class for all time but the width of the strip of analyticity may diminish as a function of time.

Finally, in Section 4 we consider $\eta \geq 0$ in (1) and complex-valued initial data in X_r-spaces for $r > 0$. Similar as in [6], where analyticity of solutions of the KdV equation was studied, we use Banach's fixed point theorem in a suitable function space in order to find a local solution of problem (1) that is analytic and has an analytic extension to a strip around the real axis although the radius of the strip of analyticity in the complex plane around the real axis may decrease with time.

Notation:

- $\hat{f} = \mathcal{F}f$: the Fourier transform of f ($\mathcal{F}^{-1}$: the inverse of the Fourier transform), where $\hat{f}(\xi) = \frac{1}{\sqrt{2\pi}} \int e^{-i\xi x} f(x)dx$ for $f \in L^1(\mathbb{R})$.
- $\|\cdot\|_s$, $(\cdot,\cdot)_s$: the norm and the inner product respectively in $H^s(\mathbb{R})$ (Sobolev space of order s of L^2 type), $s \in \mathbb{R}$. $\|f\|_s^2 \equiv \int (1+|\xi|^2)^s |\hat{f}(\xi)|^2 d\xi$.
  $\|\cdot\| = \|\cdot\|_0$: the $L^2(\mathbb{R})$ norm. $(\cdot,\cdot)$ denotes the inner product on $L^2(\mathbb{R})$.
  $H^\infty(\mathbb{R}) \equiv \cap H^s(\mathbb{R})$.
- $\mathcal{H}$: the Hilbert transform.
- $B(X,Y)$: set of bounded linear operators on X to Y. If $X = Y$ we write $B(X)$.
  $\|\cdot\|_{B(X,Y)}$: the operator norm in $B(X,Y)$.
- $S(r) = \{x + iy \in \mathbb{C};\ x \in \mathbb{R},\ |y| < r\}$, for $r > 0$.
  $A(r)$: the set of all analytic functions f on $S(r)$ such that $f \in L^2(S(r'))$ for each $0 < r' < r$ and that $f(x) \in \mathbb{R}$ for $x \in \mathbb{R}$.
- $A = (I - \partial_x^2)^{1/2}$, $X^{\sigma,s} = \mathcal{D}(A^s e^{\sigma A})$ the domain of the operator $A^s e^{\sigma A}$.
- $L^p = \{f; f \text{ is measurable on } \mathbb{R}, \|f\|_{L^p} < \infty\}$, where $\|f\|_{L^p} = \big(\int |f(x)|^p dx\big)^{1/p}$ if $1 \le p < +\infty$, and $\|f\|_{L^\infty} = \operatorname{ess\,sup}_{x\in\mathbb{R}} |f(x)|$, f is an equivalence class.
- $L_r = \{f \in L^2; \|f\|_{L_r}^2 = (f,f)_{L_r} = (\cosh(2r\xi)\hat{f}, \hat{f}) < +\infty\}$.
- $X_r = \big\{f \in L^2; \|f\|_{X_r}^2 = (f,f)_{X_r} = \sum_{j=0}^{1} (\xi^{2j}(\cosh(2r\xi) + \xi\sinh(2r\xi))\hat{f}, \hat{f}) < \infty\big\}$.
- $Y_r = \big\{f \in L^2; \partial_x f \in L^2, \|f\|_{Y_r}^2 = (f,f)_{Y_r} = \sum_{j=0}^{1} (\xi^{2j+2}\cosh(2r\xi)\hat{f}, \hat{f}) < +\infty\big\}$.
- $\|f\|_{m,p}^p = \sum_{j=0}^{m} \|\partial_x^j f\|_{L^p}^p$, $1 \le p < +\infty$. $\|f\|_{m,\infty} = \sum_{j=0}^{m} \|\partial_x^j f\|_\infty$.
- $H^p(r)$: the analytic Hardy space on the strip $S(r)$.
  $H^p(r) = \{F; F \text{ is analytic on } S(r), \|F\|_{H^p(r)} = \sup_{|y|<r} \|F(\cdot + iy)\|_{L^p} < \infty\}$.
- $H^{m,p}(r) = \{F \in H^p(r); \|F\|_{H^{m,p}(r)}^p = \sum_{j=0}^{m} \|\partial_z^j F\|_{H^p(r)}^p < \infty\}$.
- $C(I;X)$: set of continuous functions on the interval I into the Banach space X.
- $C^\omega(I;X)$: the set of weakly continuous functions from I to X.
- $\Re(z)$: the real part of the complex number z.

2. Real-valued initial data.

We deduce in this Section global analyticity (in space variables) of solutions of problems (1) and (2) when the initial data and the corresponding solution take real values on the real axis and supposing moreover that the initial data has an analytic continuation that is analytic in a strip containing the real axis. We will use the fact that problems (1) and (2) are globally well possed in $H^s(\mathbb{R})$ for $s \ge 1$, when the solution of the two previously mentioned problems take real values on the real axis at any time. More precisely we have the following two theorems (for real-valued solutions) which can be found in [1].

Theorem 2.1. *Let $s \geq 1$. If $\phi \in H^s(\mathbb{R})$, then for each $\eta > 0$ there exists a unique $u = u_\eta \in C([0,\infty); H^s(\mathbb{R}))$ solution to the problem (1) such that $\partial_t u \in C([0,\infty); H^{s-3}(\mathbb{R}))$.*

Proof. See Theorem 4.2 in [1]. $\square$

Theorem 2.2. *Let $s \geq 1$. If $\psi \in H^s(\mathbb{R})$, then for each $\eta > 0$ there exists a unique $v = v_\eta \in C([0,\infty); H^s(\mathbb{R}))$ solution to the problem (2) such that $\partial_t v \in C([0,\infty); H^{s-3}(\mathbb{R}))$.*

Proof. See Theorem 4.1 in [1]. $\square$

Theorem 2.3 (resp. 2.4) states that if the initial state has an analytic continuation that belongs to $A(r_0)$ for some $r_0 > 0$ then the solution $u(t)$ (resp. $v(t)$) of problem (1) (resp. (2)), with $\eta \geq 0$, also has an analytic continuation belonging to $A(r_1)$ for all $t \in [0,T]$, where r_1 might decrease with time. Theorem 2.3, below, is an application of the method developed by Kato&Masuda in [8] to study global analyticity (in space variables) of some partial differential equations. Similar as in the proof of Theorem 2 in [8] we consider $H^{m+5}(\mathbb{R}) \equiv Z \subset X \equiv H^{m+2}(\mathbb{R})$ and $\Phi_{\sigma;m}(v) \equiv \frac{1}{2}\|v\|^2_{\sigma,2;m}$ defined on an appropriate open set $O \subset Z$, where

$$\|f\|^2_{\sigma,s;m} \equiv \sum_{j=0}^{m} \frac{e^{2j\sigma}}{(j!)^2}\|\partial_x^j f\|_s^2 \qquad \text{and} \qquad \|f\|^2_{\sigma,s} \equiv \sum_{j=0}^{\infty} \frac{e^{2j\sigma}}{(j!)^2}\|\partial_x^j f\|_s^2, \quad s \in \mathbb{R}.$$

Lemma 2.1. *Let $F(v)$ be defined by $F(v) \equiv -vv_x - v_{xxx} - \eta(\mathcal{H}v_x + \mathcal{H}v_{xxx})$. Then there exist constants $c, \gamma > 0$ such that for every $v \in H^{m+5}(\mathbb{R})$,*

$$\langle F(v), D\Phi_{\sigma;m}(v)\rangle \leq c(\eta + \|v\|_2)\Phi_{\sigma;m}(v) + \gamma\sqrt{\Phi_{\sigma;m}(v)}\,\partial_\sigma\Phi_{\sigma;m}(v). \tag{3}$$

Proof. It is not difficult to see that $D\Phi_{\sigma;m}(v) = \sum_{j=0}^m \frac{e^{2j\sigma}}{(j!)^2}(-\partial_x)^j A^4 \partial_x^j v$, where $A = (1-\partial_x^2)^{1/2}$. Then

$$\begin{aligned}\langle F(v), D\Phi_{\sigma;m}(v)\rangle &= \sum_{j=0}^{m} \frac{e^{2j\sigma}}{(j!)^2}\Big[-\big(\partial_x^j(v\partial_x v), \partial_x^j v\big)_2 - \eta\big(\partial_x^j(\mathcal{H}\partial_x v + \mathcal{H}\partial_x^3 v), \partial_x^j v\big)_2\Big]\\ &= \sum_{j=0}^{m} \frac{e^{2j\sigma}}{(j!)^2}\Big[-\big(v\partial_x^{j+1} v, \partial_x^j v\big)_2 - Q_j(v) - \eta\big(\partial_x^j(\mathcal{H}\partial_x v + \mathcal{H}\partial_x^3 v), \partial_x^j v\big)_2\Big],\end{aligned}$$

where $Q_j(v) = \sum_{k=1}^j \binom{j}{k}\big(\partial_x^k v \partial_x^{j-k+1} v, \partial_x^j v\big)_2$, and $Q_0(v) \equiv 0$. By using Kato's inequality (K) in the Appendix we have that

$$|(v\partial_x^{j+1}v, \partial_x^j v)_2| \leq c\|\partial_x v\|_1 \|\partial_x^j v\|_2^2 \leq c\|v\|_2\|\partial_x^j v\|_2^2.$$

Moreover

$$-\big(\partial_x^j(\mathcal{H}\partial_x v+\mathcal{H}\partial_x^3 v),\partial_x^j v\big)_2 \;=\; \int(1+\xi^2)^2\xi^{2j}(|\xi|-|\xi|^3)|\hat{v}(\xi)|^2 d\xi$$
$$\leq \int(1+\xi^2)^2\xi^{2j}|\hat{v}(\xi)|^2 d\xi=\|\partial_x^j v\|_2^2. \qquad (4)$$

Then

$$\langle F(v),D\Phi_{\sigma;m}(v)\rangle\leq c(\|v\|_2+\eta)\Phi_{\sigma;m}(v)-\sum_{j=0}^{m}\frac{e^{2j\sigma}}{(j!)^2}Q_j(v). \qquad (5)$$

Now, using the Schwarz inequality and the formula $\|fg\|_2\leq\gamma\big(\|f\|_2\|g\|_1+\|f\|_1\|g\|_2\big)$ we get

$$|Q_j(v)|\leq\gamma\sum_{k=1}^{j}\binom{j}{k}\|\partial_x^j v\|_2\big(\|\partial_x^k v\|_1\|\partial_x^{j-k+1}v\|_2+\|\partial_x^k v\|_2\|\partial_x^{j-k+1}v\|_1\big). \qquad (6)$$

We denote as in [8], $b_j\equiv\frac{e^{j\sigma}}{j!}\|\partial_x^j v\|_2$, $B^2\equiv\sum_{j=0}^m b_j^2=2\Phi_{\sigma;m}(v)$ and $\tilde{B}^2\equiv\sum_{j=1}^m jb_j^2=\partial_\sigma\Phi_{\sigma;m}(v)$. By using (6) it follows, as a particular case of Lemma 3.1 in [8], that

$$\begin{aligned}\sum_{j=0}^{m}\frac{e^{2j\sigma}}{(j!)^2}|Q_j(v)| &\leq \gamma\sum_{j=1}^{m}\sum_{k=1}^{j}\big(b_j\frac{b_{k-1}}{k}(j-k+1)b_{j-k+1}+b_jb_kb_{j-k}\big)\\ &= \gamma\sum_{k=1}^{m}\frac{b_{k-1}}{k}\big(\sum_{j=k}^{m}b_j(j-k+1)b_{j-k+1}\big)+\gamma\sum_{k=1}^{m}b_k\big(\sum_{j=k}^{m}b_jb_{j-k}\big)\\ &\leq \gamma\tilde{B}^2\big(\sum_{k=1}^{m}\frac{1}{k^2}\big)^{1/2}\big(\sum_{k=1}^{m}b_{k-1}^2\big)^{1/2}+\gamma B\tilde{B}\sum_{k=1}^{m}\frac{b_k}{\sqrt{k}}\\ &\leq 2\gamma B\tilde{B}^2+\gamma B\tilde{B}\big(\sum_{k=1}^{m}\frac{1}{k^2}\big)^{1/2}\big(\sum_{k=1}^{m}kb_k^2\big)^{1/2}\leq 4\gamma B\tilde{B}^2. \end{aligned} \qquad (7)$$

By replacing the inequality (7) into (5), the Lemma follows. □

Next, we enunciate Lemma 2.4 in [8] and we give, for expository completeness, a proof of this trivial result.

Lemma 2.2. *Let $\phi_n\in A(r)$, $n=1,2,...$ be a sequence with $\|\phi_n\|_{\sigma,2}$ bounded, where $e^\sigma<r$. If $\phi_n\to 0$ in H^∞ as $n\to\infty$, then $\|\phi_n\|_{\sigma',2}\to 0$ for each $\sigma'<\sigma$.*

Proof. Let $\sigma'<\sigma$. Choose M such that $\|\phi_n\|_{\sigma,2}\leq M$ for all n. Since $\frac{e^{2j\sigma'}}{(j!)^2}\|\partial_x^j\phi_n\|_2^2\leq\frac{M^2}{e^{2j(\sigma-\sigma')}}$, it follows, by using the dominated convergence theorem, that

$$\lim_{n\to\infty}\|\phi_n\|_{\sigma',2}^2=\sum_{j=0}^{\infty}\frac{e^{2j\sigma'}}{(j!)^2}\big(\lim_{n\to\infty}\|\partial_x^j\phi_n\|_2^2\big)=0.$$

□

Theorem 2.3. *Let $\eta \geq 0$ and $T > 0$. Let $u \in C([0,T]; H^\infty(\mathbb{R}))$ be a solution of (1). If $u(0) = \phi \in A(r_0)$ for some $r_0 > 0$, there exists $r_1 > 0$ such that $u \in C([0,T]; A(r_1))$.*

Proof. Let us remark that $\phi \in A(r_0)$ implies, by Lemma 2.2 in [8], that $\phi \in H^\infty(\mathbb{R})$. So, it follows from Theorem 2.1 and from Corollary 4.7 in [7] that $u \in C([0,T]; H^\infty(\mathbb{R}))$. We proceed as in the proof of Theorem 2 in [8], to prove that $\Phi_{\sigma;m}$ is a Liapunov family for (1) on $O = Z = H^{m+5}(\mathbb{R})$, considering the functions: $\alpha(r) = \gamma\sqrt{r}$, $\beta(r) = c(\eta + M)r \equiv Kr$, for $r \geq 0$, where $M = \max_{t\in[0,T]} \|u(t)\|_2$, and $\gamma, c > 0$ are constants given by Lemma 2.1, $\rho(t) = \frac{1}{2}\|\phi\|^2_{b,2}e^{Kt}$ where $b < \sigma_0$, $e^{\sigma_0} < r_0$, and

$$\sigma(t) = b - \frac{\sqrt{2}\gamma}{K}\|\phi\|_{b,2}(e^{\frac{K}{2}t} - 1),\ t \in [0,T]. \tag{8}$$

We have that $u(t) \in A(e^{\sigma(t)})$, for all $t \in [0,T]$. Then $u(t) \in A(r_1)$ for all $t \in [0,T]$, where $r_1 = e^{\sigma(T)}$. The continuity of u, as in the proof of Theorem 2 in [8], is a consequence of Lemma 2.2 above. □

Theorem 2.4. *Let $\eta \geq 0$ and $T > 0$. Let $v \in C([0,T]; H^\infty(\mathbb{R}))$ be a solution of (2). If $v(0) = \psi \in A(r_0)$ for some $r_0 > 0$, there exists $r_1 > 0$ such that $v \in C([0,T]; A(r_1))$.*

Proof. We remark that $u \equiv v_x \in C([0,T]; H^\infty(\mathbb{R}))$ is a solution of (1) with $u(0) = \psi'$. Since $\psi' \in H^\infty(\mathbb{R})$ and $\|\psi'\|_{\sigma,0} \leq \|\psi\|_{\sigma,1}$ for each σ such that $e^\sigma < r_0$ it follows, as a consequence of Lemma 2.2 in [8], that $\psi' \in A(r_0)$. So, by Theorem 2.3, there exists $r_1 > 0$ such that $v_x \in C([0,T]; A(r_1))$. Now, since $v(t) \in H^\infty(\mathbb{R})$ and $\|v(t)\|^2_{\sigma,0} \leq \sup_{t\in[0,T]} \|v(t)\|^2 + e^{2\sigma} \sup_{t\in[0,T]} \|u(t)\|^2_{\sigma,0} < \infty$ for each σ such that $e^\sigma < r_1$ and for all $t \in [0,T]$, it follows that $v(t) \in A(r_1)$ for all $t \in [0,T]$. The continuity of v follows as in the previous Theorem. □

3. Gevrey Class Regularity.

Now, we make a Gevrey-class analysis of problems (1) and (2). Let us remark that, since each function $f \in X^{\sigma,s} = \mathcal{D}(A^s e^{\sigma A})$ with $\sigma > 0$ and $s \geq 0$, satisfies $\|f\|^2_{L_\sigma} \leq \int e^{2\sigma|\xi|}|\hat{f}(\xi)|^2 d\xi \leq \|f\|^2_{X^{\sigma,s}} < \infty$, it follows from Theorem 1 in [6] that f has an analytic extension $F \in H^2(\sigma)$, where $\sigma > 0$ is the radius of the strip of analyticity in the complex plane around the real axis. Theorem 3.1 (resp. Theorem 3.2) states that problem (1) (resp. (2)), for $\eta > 0$ and complex-valued initial data, is locally well-posed in $X^{\sigma,s}$ where $\sigma > 0$ is a fixed number and s is a suitable nonnegative number. Theorem 3.1 (resp. 3.2)) implies that if the initial condition of problem (1) (resp. (2)) is analytic and has an analytic continuation to a strip containing the real axis, then the solution of (1) (resp. (2)) has the same property, without reducing the width of the strip in time.

For $t \geq 0$ and $\xi \in \mathbb{R}$, let

$$\begin{aligned} F_\eta(t,\xi) &= e^{(i\xi^3+\eta(|\xi|-|\xi|^3))t} \\ E_\eta(t)f &= \mathcal{F}^{-1}(F_\eta(t,\cdot)\hat{f}), \;\; f \in L^2(\mathbb{R}). \end{aligned} \tag{9}$$

It is not difficult to see that $|F_\eta(t,\xi)| \leq e^{\eta t}$, for all $t \geq 0$ and $\xi \in \mathbb{R}$.

Lemma 3.1. *Let $\eta > 0$. Then $(E_\eta(t))_{t\geq 0}$ is a C^0-semigroup on $X^{\sigma,s}$ for $\sigma > 0$ and $s \in \mathbb{R}$. Moreover,*

$$\|E_\eta(t)\|_{B(X^{\sigma,s})} \leq e^{\eta t}. \tag{10}$$

When $\eta = 0$, $\|E_\eta(t)\|_{B(X^{\sigma,s})} = 1$ for all $t \geq 0$.

Proof. Similar to the proof of Lemma 2.1 in [1]. □

Lemma 3.2. *Let $t > 0$, $\lambda \geq 0$, $\eta > 0$, $\sigma > 0$ and $s \in \mathbb{R}$ be given. Then $E_\eta(t) \in B(X^{\sigma,s}, X^{\sigma,s+\lambda})$. Moreover,*

$$\|E_\eta(t)\phi\|_{X^{\sigma,s+\lambda}} \leq c_\lambda\big[e^{\eta t} + \frac{1}{(\eta t)^{\lambda/3}}\big]\|\phi\|_{X^{\sigma,s}}, \tag{11}$$

where $\phi \in X^{\sigma,s}$ and c_λ is a constant depending only on λ.

Proof.

$$\begin{aligned} \|E_\eta(t)\phi\|^2_{X^{\sigma,s+\lambda}} &\leq c_\lambda\Big[\int (1+\xi^2)^s e^{2\sigma(1+\xi^2)^{1/2}}|F_\eta(t,\xi)|^2|\hat{\phi}(\xi)|^2 d\xi \\ &\quad + \int \xi^{2\lambda}(1+\xi^2)^s e^{2\sigma(1+\xi^2)^{1/2}} e^{2\eta t(|\xi|-|\xi|^3)}|\hat{\phi}(\xi)|^2 d\xi\Big] \\ &\leq c_\lambda\big[e^{2\eta t} + \sup_{\xi\in\mathbb{R}} \xi^{2\lambda} e^{-2\eta t(|\xi|^3-|\xi|)}\big]\|\phi\|^2_{X^{\sigma,s}}. \end{aligned} \tag{12}$$

On the other hand,

$$\sup_{\xi\in\mathbb{R}} |\xi|^\lambda e^{-\eta t(|\xi|^3-|\xi|)} = \sup_{\xi\geq 0} \xi^\lambda e^{-\eta t(\xi^3-\xi)} \leq \sqrt{2}^\lambda e^{\eta t} + c_\lambda \frac{1}{(\eta t)^{\lambda/3}}. \tag{13}$$

The lemma follows immediately from (12) and (13). □

Next theorem proves, without using Bourgain-type spaces, local well-posedness to problem (1) with $\eta > 0$ in $X^{\sigma,s}$ for $\sigma > 0$, $s > 1/2$, and complex-valued initial data.

Theorem 3.1. *Let $\eta > 0$, $\sigma > 0$ and $s > 1/2$ be given. If $\phi \in X^{\sigma,s}$, then there exist $T = T_{(s,\sigma,\eta,\|\phi\|_{X^{\sigma,s}})} > 0$ and a unique function $u \in C([0,T]; X^{\sigma,s})$ satisfying the integral equation*

$$u(t) = E_\eta(t)\phi - \frac{1}{2}\int_0^t E_\eta(t-t')\partial_x(u^2(t'))dt'. \tag{14}$$

Proof. Let $M, T > 0$ be fixed but arbitrary. Let us consider the map

$$Af(t) = E_\eta(t)\phi - \frac{1}{2}\int_0^t E_\eta(t-t')\partial_x(f^2(t'))dt',$$

defined on the complete metric space

$$\Theta_{s,\sigma,\eta}(T) = \{f \in C([0,T]; X^{\sigma,s});\ \sup_{t\in[0,T]} \|f(t) - E_\eta(t)\phi\|_{X^{\sigma,s}} \le M\},$$

where $T > 0$ will be suitably chosen later. First, we prove that if $f \in \Theta_{s,\sigma,\eta}(T)$ then $Af \in C([0,T]; X^{\sigma,s})$. Without loss of generality, we may assume that $\tau > t > 0$. Then

$$\|Af(t) - Af(\tau)\|_{X^{\sigma,s}} \le \|(E_\eta(t) - E_\eta(\tau))\phi\|_{X^{\sigma,s}} + F(t,\tau) + G(t,\tau),$$

where $F(t,\tau)$ and $G(t,\tau)$ will be estimated below.

$$\begin{aligned} G(t,\tau) &\equiv \int_t^\tau \|E_\eta(\tau-t')\partial_x(f^2(t'))\|_{X^{\sigma,s}}dt' \\ &\le \frac{c_s}{\eta}(M + e^{\eta T}\|\phi\|_{X^{\sigma,s}})^2(e^{\eta(\tau-t)} - 1 + \frac{3}{2}(\eta(\tau-t))^{\frac{2}{3}}) \to 0 \ \text{ as } \tau \downarrow t, \end{aligned}$$

where in the last inequality we have used Lemmas 3.1 and 3.2 (with $\lambda = 1$) and the fact that $X^{\sigma,s}$ is a Banach algebra for $s > 1/2$ and $\sigma \ge 0$ (Lemma 6 in [2]). Since $\tau - t' \ge t - t'$, for all $t' \in [0,t]$, it follows from Lemma 3.2 and from the triangle inequality that

$$\|(E_\eta(t-t')-E_\eta(\tau-t'))\partial_x(f^2(t'))\|_{X^{\sigma,s}} \le c_s(M+e^{\eta T}\|\phi\|_{X^{\sigma,s}})^2\left(e^{\eta(T-t')}+(\eta(t-t'))^{-\frac{1}{3}}\right),$$

and the expression on the right hand side of the last inequality belongs to $L^1([0,t], dt')$. The fact that $\|(E_\eta(t-t')-E_\eta(\tau-t'))\partial_x(f^2(t'))\|_{X^{\sigma,s}} \to 0$ as $\tau \downarrow t$ is a consequence of the dominated convergence theorem. So, by using again the dominated convergence theorem, we have that

$$F(t,\tau) \equiv \int_0^t \|(E_\eta(t-t') - E_\eta(\tau-t'))\partial_x(f^2(t'))\|_{X^{\sigma,s}}dt' \to 0 \ \text{ as } \tau \downarrow t.$$

Now, we prove that $A(\Theta_{s,\sigma,\eta}(T)) \subset \Theta_{s,\sigma,\eta}(T)$, for $T = \tilde{T} > 0$ sufficiently small. Let $u \in \Theta_{s,\sigma,\eta}(T)$. Then

$$\begin{aligned} \|Au(t) - E_\eta(t)\phi\|_{X^{\sigma,s}} &\le \int_0^t \|E_\eta(t-t')\partial_x(u^2(t'))\|_{X^{\sigma,s}}dt' \\ &\le \frac{c_s}{\eta}(M + e^{\eta T}\|\phi\|_{X^{\sigma,s}})^2 h(T), \end{aligned} \tag{15}$$

where $h(T) \equiv e^{\eta T} - 1 + \frac{3}{2}(\eta T)^{2/3}$. By choosing $T = \tilde{T} > 0$ sufficiently small, the right hand side of (15) is less than M. Finally , we claim that there exists $\hat{T} \in (0, \tilde{T}]$

such that A is a contraction on $\Theta_{s,\sigma,\eta}(\hat{T})$. Let $t \in [0,\tilde{T}]$, $u, v \in \Theta_{s,\sigma,\eta}(\tilde{T})$. Then

$$\begin{aligned}
\|Au(t) - Av(t)\|_{X^{\sigma,s}} &\leq c_1 \int_0^t \big(e^{\eta(t-t')} + \frac{1}{(\eta(t-t'))^{1/3}}\big)\|\partial_x(u^2(t') - v^2(t'))\|_{X^{\sigma,s-1}} dt' \\
&\leq c_s(M + e^{\eta t}\|\phi\|_{X^{\sigma,s}}) \sup_{t'\in[0,\tilde{T}]} \|u(t') - v(t')\|_{X^{\sigma,s}} \\
&\quad \cdot \int_0^t \big(e^{\eta(t-t')} + \frac{1}{(\eta(t-t'))^{1/3}}\big) dt' \\
&\leq \frac{c_s}{\eta}(M + e^{\eta\tilde{T}}\|\phi\|_{X^{\sigma,s}})h(\tilde{T}) \sup_{t'\in[0,\tilde{T}]} \|u(t') - v(t')\|_{X^{\sigma,s}}.
\end{aligned}$$

So, by choosing $\hat{T} \in (0,\tilde{T}]$ such that $\frac{c_s}{\eta}(M+e^{\eta\hat{T}}\|\phi\|_{X^{\sigma,s}})h(\hat{T}) < 1$, the claim follows. Hence A has a unique fixed point $u \in \Theta_{s,\sigma,\eta}(\hat{T})$, which satisfies (14). Uniqueness of the solution $u \in C([0,\hat{T}]; X^{\sigma,s})$ follows from Proposition 3.2 in [1], which implies uniqueness of the solution in the class $C([0,\hat{T}]; H^s(\mathbb{R}))$ for $s > 1/2$. □

Proposition 3.1. *Problem (1) is equivalent to the integral equation (14). More precisely, if $u \in C([0,T]; X^{\sigma,s})$, $s > 1/2$ is a solution of (1) then u satisfies (14). Reciprocally, if $u \in C([0,T]; X^{\sigma,s})$, $s > 1/2$ is a solution of (14) then $u \in C^1([0,T]; X^{\sigma,s-3})$ and satisfies (1).*

Proof. Similar to the proof of Proposition 3.1 in [1]. However, here we use Lemmas 3.1 and 3.2. □

Theorem 3.2. *Let $\eta > 0$, $\sigma > 0$ and $s > 3/2$ be given. Let $\psi \in X^{\sigma,s}$. Then there exist $T = T_{(s,\sigma,\eta,\|\psi\|_{X^{\sigma,s}})} > 0$ and a unique $v \in C([0,T]; X^{\sigma,s})$ solution of (2).*

Proof. Since $\psi \in X^{\sigma,s}$, it follows that $\phi \equiv \psi' \in X^{\sigma,s-1}$. By Theorem 3.1 and Proposition 3.1, there exist $T = T_{(s,\sigma,\eta,\|\psi\|_{X^{\sigma,s}})} > 0$ and a unique $u \in C([0,T]; X^{\sigma,s-1})$ satisfying (14) and (1). Let us define

$$v(t) \equiv E_\eta(t)\psi - \frac{1}{2}\int_0^t E_\eta(t-t')u^2(t')dt', \quad t \in [0,T]. \tag{16}$$

It follows easily from (16) and from the uniqueness of the solution of (14) that $\partial_x v(t) = u(t)$. Since $u \in C([0,T]; X^{\sigma,s-1})$, it follows from Lemmas 3.1 and 3.2 that $v \in C([0,T]; X^{\sigma,s})$. Now, by similar calculations as in the proof of Proposition 3.1 in [1], we have that $v \in C^1([0,T]; X^{\sigma,s-3})$ and satisfies (2). □

Theorems 3.3 and 3.4, below, consider the case of real-valued solutions (on the real axis) to problems (1) and (2) respectively, for $\eta \geq 0$. So, Theorems 2.1 and 2.2 will be required again. The following theorem is proved similarly to Theorem 11 in [2], where the rate of decrease of the uniform radius of analyticity for KdV-type equations of the form $u_t + G(u)u_x - Lu_x = 0$ was also studied, where $u = u(x,t)$, for $x, t \in \mathbb{R}$, G is a function that is analytic at least in a neighborhood of zero in $\mathbb{C}$, but real-valued on the real axis, and L is a homogeneus Fourier multiplier operator defined by $\widehat{Lu}(\xi) = |\xi|^{\mu}\hat{u}(\xi)$, for some $\mu > 0$. In the case of problem (1) we have that $\widehat{Lu}(\xi) = [\xi^2 - \eta i(\text{sgn}(\xi) - \xi|\xi|)]\hat{u}(\xi)$.

Theorem 3.3. *Let $\eta \geq 0$ and $T > 0$. Suppose that $\phi \in X^{\sigma_0,s}$, for some $\sigma_0 > 0$ and $s > 5/2$. Suppose moreover that $\phi(x) \in \mathbb{R}$ for $x \in \mathbb{R}$. Then the solution u of problem (1) satisfies $u \in C([0,T]; X^{\sigma(T),s})$, where $\sigma(t)$ is a positive monotone decreasing function given by (24).*

Proof. Let η, T, σ_0, and s be as in the hypothesis of the theorem. Let $r \equiv s-1 > 3/2$. By the Remark at the end of this Section we have that $\phi \in A(\sigma_0)$. So, by using Lemma 2.2 in [8], we have that $\phi \in H^{\infty}(\mathbb{R})$. Then $u \in C([0,T]; H^s(\mathbb{R}))$, which follows from Theorem 2.1 and from Corollary 4.7 in [7]. Let $v \equiv u_x$, then

$$v_t + v^2 + v_{xxx} + \eta(\mathcal{H}v_x + \mathcal{H}v_{xxx}) + uv_x = 0, \;\; v(0) = \phi'. \tag{17}$$

Let $\sigma \in C^1([0,T]; \mathbb{R})$ be a positive function such that $\sigma' < 0$ and $\sigma(0) = \sigma_0$. Then

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|^2_{X^{\sigma(t),r}} - \sigma'(t)\|v(t)\|^2_{X^{\sigma(t),r+1/2}} = \Re\int(1+\xi^2)^r e^{2\sigma(t)(1+\xi^2)^{1/2}}\partial_t\hat{v}(t,\xi)\overline{\hat{v}(t,\xi)}d\xi.$$

It follows from the last expression and from (17) that

$$\frac{1}{2}\frac{d}{dt}\|v(t)\|^2_{X^{\sigma(t),r}} - \sigma'(t)\|v(t)\|^2_{X^{\sigma(t),r+1/2}} \leq I_1 + I_2 + \eta\|v(t)\|^2_{X^{\sigma(t),r}}, \tag{18}$$

where

$$\begin{cases} I_1 \equiv |(A^r e^{\sigma(t)A}v^2(t), A^r e^{\sigma(t)A}v(t))|, \\ I_2 \equiv |(A^r e^{\sigma(t)A}(uv_x)(t), A^r e^{\sigma(t)A}v(t))|. \end{cases}$$

I_1 and I_2 are particular cases of the corresponding ones in [2], taking $G(u) = u$. For the sake of completeness we estimate them here. Since $r > 1/2$, by using Lemmas 6 and 9 in [2], we have that

$$\begin{aligned} I_1 &\leq c_r\|A^r e^{\sigma(t)A}v(t)\|^3 \leq c_r\|A^r v(t)\|^3 + \tilde{c}_r\sigma(t)\|A^{r+1/3}e^{\sigma(t)A}v(t)\|^3 \\ &\leq c_r\|A^r v(t)\|^3 + \tilde{c}_r\sigma(t)\|A^r e^{\sigma(t)A}v(t)\|\|A^{r+1/2}e^{\sigma(t)A}v(t)\|^2. \end{aligned} \tag{19}$$

Since $r > 3/2$, by using Lemma 10 in [2], we obtain

$$I_2 \leq c_r\|A^{r+1}u(t)\|\|A^r v(t)\|^2 + \tilde{c}_r\sigma(t)\|A^{r+1}e^{\sigma(t)A}u(t)\|\|A^{r+1/2}e^{\sigma(t)A}v(t)\|^2. \tag{20}$$

Since $u \in C([0,T]; H^s(\mathbb{R}))$, it follows that $\|A^{r+1}u(t)\| \leq c_{(r,T)}$. Moreover

$$\begin{aligned} &\|A^{r+1}e^{\sigma(t)A}u(t)\|^2 = \int(1+\xi^2)^r e^{2\sigma(t)(1+\xi^2)^{1/2}}|\hat{u}(\xi,t)|^2 d\xi + \|A^r e^{\sigma(t)A}v(t)\|^2 \\ &\leq e^{2\sqrt{2}\sigma_0}\|u(t)\|^2_r + 2\|A^r e^{\sigma(t)A}v(t)\|^2 \leq c_{(r,T,\sigma_0)} + 2\|A^r e^{\sigma(t)A}v(t)\|^2, \end{aligned} \tag{21}$$

where $c_{(r,T,\sigma_0)}$ is a positive constant depending only on r, T and σ_0. Replacing the last inequalities into (20) and since $v \in C([0,T]; H^r(\mathbb{R}))$, we get

$$I_2 \leq c_{(r,T)} + \tilde{c}_r \sigma(t) \big(c_{(r,T,\sigma_0)} + \sqrt{2} \| A^r e^{\sigma(t)A} v(t) \| \big) \| A^{r+1/2} e^{\sigma(t)A} v(t) \|^2. \tag{22}$$

Let us remark that $c_{(r,T)}$ and $c_{(r,T,\sigma_0)}$ are positive, continuous, non-decreasing functions of the variable $T \in [0, +\infty)$. Now, by using (19) and (22) into (18), we obtain

$$\begin{aligned} &\frac{1}{2} \frac{d}{dt} \| v(t) \|^2_{X^{\sigma(t),r}} - \sigma'(t) \| v(t) \|^2_{X^{\sigma(t),r+1/2}} \\ &\leq c_{(r,T)} + c_{(r,T,\sigma_0)} \sigma(t) \big(1 + \| v(t) \|_{X^{\sigma(t),r}} \big) \| v(t) \|^2_{X^{\sigma(t),r+1/2}} + \eta \| v(t) \|^2_{X^{\sigma(t),r}}. \end{aligned}$$

Then

$$\begin{aligned} &\frac{d}{dt} \| v(t) \|^2_{X^{\sigma(t),r}} + 2 \big(-\sigma'(t) - c_{(r,T,\sigma_0)} \sigma(t) (1 + \| v(t) \|_{X^{\sigma(t),r}}) \big) \| v(t) \|^2_{X^{\sigma(t),r+1/2}} \\ &\leq c_{(r,T)} + 2\eta \| v(t) \|^2_{X^{\sigma(t),r}}. \end{aligned} \tag{23}$$

Inequality (23) implies that $v(t) \in X^{\sigma(t),r}$ for all $t \in [0,T]$, where

$$\sigma(t) = \sigma_0 e^{-Kt}, \tag{24}$$

$K = c_{(r,T,\sigma_0)} (1 + c_{(r,T,\sigma_0,\phi)} e^{\eta T})$, and $c_{(r,T,\sigma_0,\phi)} = \big(\| A^r e^{\sigma_0 A} \phi' \|^2 + c_{(r,T)} T \big)^{1/2}$. More precisely we have that

$$\| v(t) \|_{X^{\sigma(t),r}} \leq c_{(r,T,\sigma_0,\phi)} e^{\eta T}, \ t \in [0,T]. \tag{25}$$

Now, we will prove assertion (25). In fact let

$$T^* \equiv \sup \{ T > 0; \exists!\, u \in C([0,T]; X^{\sigma(T),s}) \text{ solution of (1)}, \sup_{t \in [0,T]} \| u(t) \|_{X^{\sigma(t),s}} < \infty \}.$$

We claim that $T^* = +\infty$. Suppose by contradiction that $T^* < \infty$. It follows from Theorem 3.1, Proposition 3.1 and Theorem 1 in [5] (for $\eta = 0$) that $T^* > 0$. Let $\tilde{T} < T^*$. We have that

$$\sup_{t \in [0,\tilde{T}]} \| v(t) \|_{X^{\sigma(t),r}} \leq \sup_{t \in [0,\tilde{T}]} \| u(t) \|_{X^{\sigma(t),s}} \equiv M_{(\tilde{T})} \equiv M.$$

By choosing $\sigma(t) = \sigma_0 e^{-\tilde{K} t}$ for $t \in [0, \tilde{T}]$, where $\tilde{K} \equiv c_{(r,\tilde{T},\sigma_0)} (1 + M)$, we have that

$$-\sigma'(t) - c_{(r,\tilde{T},\sigma_0)} \sigma(t) (1 + \| v(t) \|_{X^{\sigma(t),r}}) = \sigma(t) c_{(r,\tilde{T},\sigma_0)} (M - \| v(t) \|_{X^{\sigma(t),r}}) \geq 0,$$

for all $t \in [0, \tilde{T}]$. So, it follows from (23) that

$$\frac{d}{dt} \| v(t) \|^2_{X^{\sigma(t),r}} \leq c_{(r,\tilde{T})} + 2\eta \| v(t) \|^2_{X^{\sigma(t),r}},$$

for all $t \in [0, \tilde{T}]$. Now, Gronwall´s inequality implies that $\| v(t) \|_{X^{\sigma(t),r}} \leq c_{(r,\tilde{T},\sigma_0,\phi)} e^{\eta t}$, for all $t \in [0, \tilde{T}]$, where $c_{(r,\tilde{T},\sigma_0,\phi)} \equiv (\| \phi' \|^2_{X^{\sigma_0,r}} + c_{(r,\tilde{T})} \tilde{T})^{1/2}$ is a continuous, positive, non-decreasing function of $\tilde{T} \in [0, \infty)$. So, we can replace above the upper bound M by $c_{(r,\tilde{T},\sigma_0,\phi)} e^{\eta \tilde{T}}$ and take $\sigma(t) = \sigma_0 e^{-Kt}$, $K \equiv c_{(r,\tilde{T},\sigma_0)} (1 + c_{(r,\tilde{T},\sigma_0,\phi)} e^{\eta \tilde{T}})$, for $t \in [0, \tilde{T}]$. Since $c_{(r,\tilde{T},\sigma_0)}$ and $c_{(r,\tilde{T},\sigma_0,\phi)}$ are continuous, non-decreasing functions of

$\tilde{T} \in [0,\infty)$, we can choose $\sigma(t) = \sigma_0 e^{-\hat{K}t}$, where $\hat{K} \equiv c_{(r,T^*,\sigma_0)}(1 + c_{(r,T^*,\sigma_0,\phi)}e^{\eta T^*})$ for all $t \in [0, T^*]$, and applying again the local theory we obtain a contradiction. Finally, it follows from (21) and (25) that $u(t) \in X^{\sigma(t),s} \subset X^{\sigma(T),s}$ for all $t \in [0, T]$. □

Theorem 3.4. *Let $\eta \geq 0$ and $T > 0$. Suppose that $\psi \in X^{\sigma_0,s}$, for some $\sigma_0 > 0$ and $s > 7/2$. Suppose moreover that $\psi(x) \in \mathbb{R}$ for $x \in \mathbb{R}$. Then the solution v of problem (2) satisfies $v \in C([0,T]; X^{\sigma(T),s})$, where $\sigma(t)$ is a positive monotone decreasing function given by (24).*

Proof. Since $\psi' \in X^{\sigma_0,s-1}$, and $s - 1 > 5/2$, it follows from Theorem 3.3 that $u \equiv v_x \in C([0,T]; X^{\sigma(T),s-1})$, where σ is given by (24). Making similar calculations to (21), we see that $v(t) \in X^{\sigma(t),s}$ for all $t \in [0, T]$. Finally, since

$$\|v(t) - v(\tau)\|^2_{X^{\sigma(T),s}} \leq e^{2\sqrt{2}\sigma(T)}\|v(t) - v(\tau)\|^2_{s-1} + \|u(t) - u(\tau)\|^2_{X^{\sigma(T),s-1}},$$

it follows that $v \in C([0,T]; X^{\sigma(T),s})$. □

Remark: For $r > 0$, we have

$$\begin{cases} X^{r,s} \subset L_r, & \text{if} \quad s \geq 0. \\ X^{r,s} \subset X_r, & \text{if} \quad s \geq 3/2. \\ X_r \subset L_r. \\ \text{If } \ f \in L_r, \ \text{ and } \quad f(x) \in \mathbb{R} \ \text{ for } \ x \in \mathbb{R}, \ \text{ then } \ f \in A(r). \end{cases} \tag{26}$$

The first statement above was already proved at the beginning of this Section. The second part in (26) follows from the inequality $\|f\|^2_{X_r} \leq 4\|f\|^2_{X^{r,s}}$ for $s \geq 3/2$. The fact that $X_r \subset L_r$ is obvious. Finally, suppose that $f \in L_r$ and $f(x) \in \mathbb{R}$ for $x \in \mathbb{R}$; then we already know (see the first paragraph of this Section) that f has an analytic extension $F \in H^2(r)$; moreover, for every $0 < r' < r$ we have

$$\int_{-r'}^{r'}\int_{-\infty}^{+\infty} |F(x+iy)|^2 dxdy \leq \int_{-r'}^{r'} \big(\sup_{|y|<r'} \int_{-\infty}^{+\infty} |F(x+iy)|^2 dx \big) dy \leq 2r\|F\|^2_{H^2(r)}.$$

4. Analyticity of Local Solutions of (1) in X_r-Spaces.

In this section we show that if the initial data ϕ of (1), with $\eta \geq 0$, is analytic and has an analytic continuation to a strip containing the real axis, then there exists a $T > 0$ such that the solution $u(t)$ of (1) has the same property for all $t \in [0, T]$, but the width of the strip may decrease as a function of time. Similar to the proof of Theorem 2 in [6], we establish in Theorem 4.1 existence and analyticity of the solution of problem (1) simultaneously including the case when the initial condition is complex-valued on the real axis, more precisely when $\phi \in X_{\sigma_0}$ for some $\sigma_0 > 0$. The following lemma, which states a close relation between spaces X_r and Y_r, will be mainly used to prove that $u \in C^\omega([0,T]; X_{\sigma(T)})$ in Theorem 4.1.

Lemma 4.1. *Y_r is a dense subset of X_r, for $r > 0$.*

Proof. Let $f \in Y_r$. We have that

$$\begin{aligned}
\int \cosh(2r\xi)|\hat{f}(\xi)|^2 d\xi &\le \cosh(2r)\|f\|^2 + \int_{|\xi|>1} \cosh(2r\xi)|\hat{f}(\xi)|^2 d\xi \\
&\le \cosh(2r)(\|f\|^2 + \|f\|^2_{Y_r}), \\
\int \xi \sinh(2r\xi)|\hat{f}(\xi)|^2 d\xi &\le \sinh(2r)\|f\|^2 + \int_{|\xi|>1} \xi^2 \cosh(2r\xi)|\hat{f}(\xi)|^2 d\xi \\
&\le \cosh(2r)(\|f\|^2 + \|f\|^2_{Y_r}),
\end{aligned}$$

and similarly $\int \xi^3 \sinh(2r\xi)|\hat{f}(\xi)|^2 d\xi \le \cosh(2r)(\|f\|^2 + \|f\|^2_{Y_r})$. So, using the last three inequalities, it is not difficult to prove that

$$\|f\|^2_{X_r} \le 4\cosh(2r)(\|f\|^2 + \|f\|^2_{Y_r}). \tag{27}$$

It follows from the last inequality that $Y_r \subset X_r$. Now, let us consider the set

$$L_{2,0}(\mathbb{R}) = \big\{g \in L^2(\mathbb{R});\ \exists K > 0,\ |\hat{g}(\xi)| \le K \text{ a.e. in } \mathbb{R}, \hat{g}(\xi) = 0 \text{ a.e. in } \{\xi; |\xi| > K\}\big\}.$$

Let $g \in L_{2,0}(\mathbb{R})$, then $\sum_{j=0}^{1} \int \xi^{2j+2} \cosh(2r\xi)|\hat{g}(\xi)|^2 d\xi \le (K^2 + K^4)\cosh(2rK)\|g\|^2$, so $g \in Y_r$. Now we will prove that $L_{2,0}(\mathbb{R})$ is a dense set in X_r. Let f be an element of X_r. For each $n \in \mathbb{N}$, take $f_n \in L_{2,0}(\mathbb{R})$ given by

$$\widehat{f_n}(\xi) = \begin{cases} \hat{f}(\xi), & \text{if } |\xi| \le n \text{ and } |\hat{f}(\xi)| \le n \\ 0, & \text{otherwise.} \end{cases}$$

It follows easily from the definition of f_n that $|\widehat{f_n}(\xi)| \le |\hat{f}(\xi)|$, for all $n \in \mathbb{N}$ and $\xi \in \mathbb{R}$. Moreover, $\widehat{f_n}(\xi) \to \hat{f}(\xi)$ as $n \to \infty$, for all $\xi \in \mathbb{R}$. So, by the dominated convergence theorem, we have that $\|f_n - f\|_{X_r} \to 0$ as $n \to \infty$. This concludes the proof. □

Lemma 4.2. *(i.) Suppose that $F \in H^{2,2}(r)$. Let f be the trace of F on the real line. Then $f \in Y_r$ and*

$$\|f\|_{Y_r} \le \sqrt{2}\|F\|_{H^{2,2}(r)}. \tag{28}$$

(ii.) Conversely, suppose that $f \in Y_r$. Then f has an analytic extension $F \in H^{2,2}(r)$ and

$$\|F\|_{H^{2,2}(r)} \le \sqrt{10\cosh(2r)}(\|f\| + \|f\|_{Y_r}). \tag{29}$$

Proof. (i.) Since $F \in H^{2,2}(r)$, we see that $\partial_z F, \partial_z^2 F \in H^2(r)$. Then, by using Theorem 1 in [6], we have that $f', f'' \in L_r$ and moreover

$$\|f\|^2_{Y_r} = \|f'\|^2_{L_r} + \|f''\|^2_{L_r} \le 2(\|\partial_z F\|^2_{H^2(r)} + \|\partial_z^2 F\|^2_{H^2(r)}) \le 2\|F\|^2_{H^{2,2}(r)}.$$

(ii.) By Lemma 4.1, $Y_r \subset X_r$. So, it follows from Lemma 2.1 in [6] that f has an analytic extension F on $S(r)$ such that $\|\partial_z F\|_{H^{1,2}(r)} \le \sqrt{2}\|f\|_{Y_r}$ and $\|F\|_{H^{1,2}(r)} \le \sqrt{2}\|f\|_{X_r}$. Now, using the last inequalities, we have that

$$\begin{aligned} \|F\|^2_{H^{2,2}(r)} &= \|F\|^2_{H^{1,2}(r)} + \|\partial_z^2 F\|^2_{H^2(r)} \le \|F\|^2_{H^{1,2}(r)} + \|\partial_z F\|^2_{H^{1,2}(r)} \\ &\le 2(\|f\|^2_{X_r} + \|f\|^2_{Y_r}) \le 10\cosh(2r)(\|f\|^2 + \|f\|^2_{Y_r}), \end{aligned}$$

where in the last inequality we have used (27). □

Lemma 4.3. *Let $r > 0$. Suppose that $F, G \in H^{2,2}(r)$. Then $FG \in H^{2,2}(r)$ and*

$$\|FG\|_{H^{2,2}(r)} \le c\|F\|_{H^{2,2}(r)}\|G\|_{H^{2,2}(r)}. \tag{30}$$

Proof. Using Leibniz's rule and Sobolev's inequality ($\|G\|_{H^\infty(r)} \le c\|G\|_{H^{1,2}(r)}$) we have that

$$\begin{aligned} \|\partial_z^2(FG)\|_{H^2(r)} &\le \|\partial_z^2 F\|_{H^2(r)}\|G\|_{H^\infty(r)} + 2\|\partial_z F\|_{H^\infty(r)}\|\partial_z G\|_{H^2(r)} \\ &\quad + \|F\|_{H^\infty(r)}\|\partial_z^2 G\|_{H^2(r)}. \\ &\le c\|F\|_{H^{2,2}(r)}\|G\|_{H^{2,2}(r)}. \end{aligned}$$

Moreover, by Lemma 2.3 in [6], we have that $\|FG\|_{H^{1,2}(r)} \le c\|F\|_{H^{1,2}(r)}\|G\|_{H^{1,2}(r)}$. Since $\|FG\|^2_{H^{2,2}(r)} = \|FG\|^2_{H^{1,2}(r)} + \|\partial_z^2(FG)\|^2_{H^2(r)}$, the proof of the Lemma follows from the last two inequalities. □

Lemma 4.4 and Corollary 4.1, below, are particular cases of Lemma 2.4 and its Corollary in Hayashi [6] respectively.

Lemma 4.4. *There exists a polynomial $\tilde{a}$ of which coefficients are all nonnegative with the following property: If $r > 0$ and $f, g, v \in X_r \cap Y_r$, then*

$$|(f\partial_x f - g\partial_x g, v)_{X_r}| \le \tilde{a}(\|f\|_{X_r}, \|g\|_{X_r})\big(\|f\|_{Y_r}\|f-g\|_{X_r} + \|f-g\|_{Y_r}\big)\big(\|v\|_{X_r} + \|v\|_{Y_r}\big).$$

Corollary 4.1. *There exists a polynomial $\tilde{a_1}$ with nonnegative coefficients such that if $f, v \in X_r \cap Y_r$, then*

$$|(f\partial_x f, v)_{X_r}| \le \tilde{a_1}(\|f\|_{X_r})\|f\|_{Y_r}\big(\|v\|_{X_r} + \|v\|_{Y_r}\big). \tag{31}$$

Now, we state the main theorem of this Section.

Theorem 4.1. *Let $\eta \ge 0$. If $\phi \in X_{\sigma_0}$ for some $\sigma_0 > 0$, then there exist a $T = T(\|\phi\|_{X_{\sigma_0}}, \eta, \sigma_0) > 0$ and a positive monotone decreasing function $\sigma(t)$ satisfying $\sigma(0) = \sigma_0$ and such that (1) has a unique solution $u \in C^\omega([0,T]; X_{\sigma(T)})$. When $\eta = 0$, $u \in C([0,T]; X_{\sigma(T)})$.*

Proof. Let $\eta \geq 0$, $\sigma_0 > 0$, and $\phi \in X_{\sigma_0}$. Let $\sigma(t) = \sigma_0 e^{-At/\sigma_0}$, where the positive constant A will be conveniently chosen later. For $T > 0$, consider the space

$$B(T) = \big\{f : [0,T]\times\mathbb{R} \mapsto \mathbb{C}; \|f\|^2_{B(T)} = \sup_{0\leq t\leq T} \|f(t)\|^2_{X_{\sigma(t)}} + A\int_0^T \|f(t)\|^2_{Y_{\sigma(t)}} dt < +\infty\big\}$$

and

$$B_\rho(T) = \{f \in B(T); \|f\|_{B(T)} \leq \rho\}.$$

Let $\rho = 4\|\phi\|_{X_{\sigma_0}}$. For $v \in B_\rho(T)$, we define the mapping M by $u = Mv$, where u is the solution of the linearized problem

$$\left\{\begin{array}{l} \partial_t u + \partial_x^3 u + \eta(\mathcal{H}\partial_x u + \mathcal{H}\partial_x^3 u) = -v\partial_x v \\ u(0) = \phi. \end{array}\right. \tag{32}$$

More precisely u can be obtained as follows. Take the Fourier transform to (32), then

$$\partial_t \hat{u}(t,\xi) - i\xi^3\hat{u}(t,\xi) + \eta(-|\xi| + |\xi|^3)\hat{u}(t,\xi) = -\widehat{vv_x}(t,\xi).$$

Now, integrating the last expression between 0 and t, it follows that

$$\hat{u}(t,\xi) = e^{(i\xi^3+\eta(|\xi|-|\xi|^3))t}\hat{\phi}(\xi) - \int_0^t e^{(i\xi^3+\eta(|\xi|-|\xi|^3))(t-\tau)}\widehat{vv_x}(\tau,\xi)d\tau, \tag{33}$$

where the last integral is well defined, since

$$\begin{aligned} \int_0^t |\widehat{vv_x}(\tau,\xi)|d\tau &= \int_0^t \big|\big(\widehat{v(\tau)} * \widehat{v_x(\tau)}\big)(\xi)\big|d\tau = \int_0^t \Big|\int \hat{v}(\tau,\xi-\omega)\omega\hat{v}(\tau,\omega)d\omega\Big|d\tau \\ &\leq \int_0^t \|v(\tau)\|\|v(\tau)\|_1 d\tau \leq \rho^2 t, \end{aligned}$$

and in the last inequality we have used the fact that

$$\|v(\tau)\|_1 \leq \|v(\tau)\|_{X_{\sigma(\tau)}} \leq \|v\|_{B(T)} \leq \rho.$$

Let us choose $A, T > 0$ such that

$$e^{-AT/\sigma_0} > \frac{1}{2}. \tag{34}$$

Let $v \in B_\rho(T)$. It will be proved that $u = Mv \in B_\rho(T)$, for suitably chosen $T, A > 0$. Now, it is not difficult to see that

$$\begin{aligned} \frac{d}{dt}\|u(t)\|^2_{X_{\sigma(t)}} &= 2\sigma'(t)\sum_{j=0}^1 \int \xi^{2j}\big(\xi\sinh(2\sigma(t)\xi) + \xi^2\cosh(2\sigma(t)\xi)\big)|\hat{u}(t,\xi)|^2 d\xi \\ &\quad + 2\Re\sum_{j=0}^1 \int \xi^{2j}\big(\cosh(2\sigma(t)\xi) + \xi\sinh(2\sigma(t)\xi)\big)\partial_t\hat{u}(t,\xi)\overline{\hat{u}(t,\xi)}d\xi. \end{aligned}$$

Taking the Fourier transform to (32), multiplying both sides of the obtained equation by $\xi^{2j}\big(\cosh(2\sigma(t)\xi)+\xi\sinh(2\sigma(t)\xi)\big)\overline{\hat{u}(t,\xi)}$, integrating with respect to ξ, summing the terms for $j=0,1$ and finally taking the real part we get

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{X_{\sigma(t)}}-\sigma'(t)\sum_{j=0}^{1}\int\xi^{2j}\big(\xi\sinh(2\sigma(t)\xi)+\xi^2\cosh(2\sigma(t)\xi)\big)|\hat{u}(t,\xi)|^2d\xi$$
$$+\eta\sum_{j=0}^{1}\int\xi^{2j}\big(\cosh(2\sigma(t)\xi)+\xi\sinh(2\sigma(t)\xi)\big)(-|\xi|+|\xi|^3)|\hat{u}(t,\xi)|^2d\xi$$
$$=-\Re(vv_x,u)_{X_{\sigma(t)}}.$$

Since $\sigma'(t)=-Ae^{-\frac{At}{\sigma_0}}$ and $|\xi|-|\xi|^3\le 1$, for all $\xi\in\mathbb{R}$, it follows that

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{X_{\sigma(t)}}+Ae^{-\frac{At}{\sigma_0}}\|u(t)\|^2_{Y_{\sigma(t)}}\le-\Re(vv_x,u)_{X_{\sigma(t)}}+\eta\|u(t)\|^2_{X_{\sigma(t)}}.$$

Using (34) and Corollary 4.1, it follows from the last inequality that

$$\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{X_{\sigma(t)}}+\frac{A}{2}\|u(t)\|^2_{Y_{\sigma(t)}}\le a(\rho)\|v(t)\|_{Y_{\sigma(t)}}(\|u(t)\|_{X_{\sigma(t)}}+\|u(t)\|_{Y_{\sigma(t)}})+\eta\|u(t)\|^2_{X_{\sigma(t)}},$$

where $a(\cdot)$ is a polynomial with nonnegative coefficients. Now integrating the last inequality from 0 to t we get

$$\sup_{0\le t\le T}\|u(t)\|^2_{X_{\sigma(t)}}+A\int_0^T\|u(t)\|^2_{Y_{\sigma(t)}}dt\le 2\big[\|\phi\|^2_{X_{\sigma_0}}+2a(\rho)\big(\int_0^T\|v(t)\|^2_{Y_{\sigma(t)}}dt\big)^{1/2}$$
$$\cdot\big[\big(\int_0^T\|u(t)\|^2_{X_{\sigma(t)}}dt\big)^{1/2}+\big(\int_0^T\|u(t)\|^2_{Y_{\sigma(t)}}dt\big)^{1/2}\big]+2\eta\int_0^T\|u(t)\|^2_{X_{\sigma(t)}}dt\big].$$

Then,

$$\begin{aligned}\|u\|^2_{B(T)}&\le\frac{\rho^2}{8}+\frac{4\rho a(\rho)}{\sqrt{A}}(\sqrt{T}+\frac{1}{\sqrt{A}})\|u\|_{B(T)}+4\eta T\|u\|^2_{B(T)}\\&\le\frac{\rho^2}{8}+\frac{1}{2}\big[\frac{4\rho a(\rho)}{\sqrt{A}}(\sqrt{T}+\frac{1}{\sqrt{A}})\big]^2+(\frac{1}{2}+4\eta T)\|u\|^2_{B(T)}.\end{aligned}$$

By choosing $T>0$ small enough such that

$$\eta T<\frac{1}{12},\tag{35}$$

we have that

$$\|u\|^2_{B(T)}\le(\frac{3}{4}+\frac{48a^2(\rho)}{A}(\sqrt{T}+\frac{1}{\sqrt{A}})^2)\rho^2.$$

Now we take $A,T>0$ such that

$$\frac{a(\rho)}{\sqrt{A}}(\sqrt{T}+\frac{1}{\sqrt{A}})<\frac{1}{8\sqrt{3}}.\tag{36}$$

So, choosing $A,T>0$ such that (34)-(36) are satisfied, it follows that $u=Mv\in B_\rho(T)$.

Now let us prove that the mapping M is a contraction, defined from $B_\rho(T)$ into itself. Let $v_1, v_2 \in B_\rho(T)$, $w = Mv_1 - Mv_2 = u_1 - u_2$, where $u_1(0) = u_2(0) = \phi$. Then,

$$\partial_t \hat{w}(t,\xi) - i\xi^3 \hat{w}(t,\xi) + \eta(-|\xi| + |\xi|^3)\hat{w}(t,\xi) = \mathcal{F}(-v_1\partial_x v_1 + v_2\partial_x v_2)(t,\xi).$$

Multiplying both sides of the last equation by $\xi^{2j}\big(\cosh(2\sigma(t)\xi)+\xi\sinh(2\sigma(t)\xi)\big)\overline{\hat{w}(t,\xi)}$, integrating in ξ, summing the terms for $j = 0, 1$ and taking the real part we obtain

$$\begin{aligned}\frac{d}{dt}\|w(t)\|^2_{X_{\sigma(t)}} + A\|w(t)\|^2_{Y_{\sigma(t)}} &\le -2\Re(v_1\partial_x v_1 - v_2\partial_x v_2, w)_{X_{\sigma(t)}} + 2\eta\|w(t)\|^2_{X_{\sigma(t)}}\\ &\le 2\tilde{a}(\rho,\rho)\big(\|v_1(t)\|_{Y_{\sigma(t)}}\|v_1 - v_2\|_{B(T)} + \|v_1(t) - v_2(t)\|_{Y_{\sigma(t)}}\big)\\ &\quad\cdot\big(\|w(t)\|_{X_{\sigma(t)}} + \|w(t)\|_{Y_{\sigma(t)}}\big) + 2\eta\|w(t)\|^2_{X_{\sigma(t)}},\end{aligned}$$

where the last inequality is a consequence of Lemma 4.4 and $\tilde{a}(\cdot,\cdot)$ is a polynomial with nonnegative coefficients. Integrating the last expression on $[0,t]$ and applying the Cauchy Schwarz inequality, we get

$$\begin{aligned}&\|w\|^2_{B(T)} \le 4\tilde{a}(\rho,\rho)\big[\|v_1 - v_2\|_{B(T)}\big(\int_0^T \|v_1(t)\|^2_{Y_{\sigma(t)}}dt\big)^{\frac{1}{2}} + \big(\int_0^T \|v_1(t) - v_2(t)\|^2_{Y_{\sigma(t)}}dt\big)^{\frac{1}{2}}\big]\\ &\cdot\big[\big(\int_0^T \|w(t)\|^2_{X_{\sigma(t)}}dt\big)^{\frac{1}{2}} + \big(\int_0^T \|w(t)\|^2_{Y_{\sigma(t)}}dt\big)^{\frac{1}{2}}\big] + 4\eta T\|w\|^2_{B(T)}\\ &\le \frac{4\tilde{a}(\rho,\rho)}{\sqrt{A}}(\|v_1\|_{B(T)} + 1)\|v_1 - v_2\|_{B(T)}\big(\sqrt{T} + \frac{1}{\sqrt{A}}\big)\|w\|_{B(T)} + 4\eta T\|w\|^2_{B(T)}.\end{aligned}$$

By using (35) in the last inequality we obtain

$$\|w\|_{B(T)} \le \frac{6\tilde{a}(\rho,\rho)}{\sqrt{A}}(1+\rho)\big(\sqrt{T} + \frac{1}{\sqrt{A}}\big)\|v_1 - v_2\|_{B(T)}.$$

If we take $A, T > 0$ such that

$$\frac{6\tilde{a}(\rho,\rho)}{\sqrt{A}}(1+\rho)\big(\sqrt{T} + \frac{1}{\sqrt{A}}\big) < 1, \tag{37}$$

then M is a contraction. So, by choosing $A, T > 0$ such that (34)-(37) are satisfied, the mapping M has a unique fixed point $u \in B_\rho(T)$ that is the solution of problem (1). Since $\|u(t)\|_{X_{\sigma(T)}} \le \|u(t)\|_{X_{\sigma(t)}}$, for all $t \in [0,T]$, then $u(t) \in X_{\sigma(T)}$, for all $t \in [0,T]$.

Now we will prove that $u \in C^\omega([0,T]; X_{\sigma(T)})$. Let $t \in [0,T]$, without loss of generality we may assume $0' \le t' < t \le T$. Since by Lemma 4.1, $Y_{\sigma(T)}$ is a dense subset of $X_{\sigma(T)}$, it is enough to prove that $(u(t) - u(t'), f)_{X_{\sigma(T)}} \to 0$ as $t' \uparrow t$, for all $f \in Y_{\sigma(T)}$. So, let f be an arbitrary but fixed element of $Y_{\sigma(T)}$. Let us denote by $h_j(\xi,T) \equiv \xi^{2j}\big(\cosh(2\sigma(T)\xi) + \xi\sinh(2\sigma(T)\xi)\big)$. First, let us remark that $|\hat{u}(t,\xi) - \hat{u}(t',\xi)| \to 0$ as $t' \uparrow t$, for all $\xi \in \mathbb{R}$. In fact, by using (33), we have that $\hat{u}(t,\xi) - \hat{u}(t',\xi) = \sum_{j=1}^3 I_j(t,t',\xi)$, where

$$I_1(t,t',\xi) \equiv \big(F_\eta(t,\xi) - F_\eta(t',\xi)\big)\hat{\phi}(\xi) \longrightarrow 0 \text{ as } t' \uparrow t,$$

$$\begin{aligned}|I_2(t,t',\xi)| &\equiv \Big|\int_{t'}^{t} F_\eta(t-\tau,\xi)\widehat{uu_x}(\tau,\xi)d\tau\Big| \le \frac{e^{\eta T}|\xi|}{2}\int_{t'}^{t}|\widehat{u(\tau)^2}(\xi)|d\tau \\ &\le \frac{e^{\eta T}|\xi|}{2\sqrt{2\pi}}\int_{t'}^{t}\|u(\tau)\|^2 d\tau \le ce^{\eta T}|\xi|\|u\|^2_{B(T)}(t-t') \longrightarrow 0 \text{ as } t'\uparrow t,\end{aligned}$$

$$|I_3(t,t',\xi)| \equiv \Big|\int_0^{t'}\big(F_\eta(t-\tau,\xi)-F_\eta(t'-\tau,\xi)\big)\widehat{uu_x}(\tau,\xi)d\tau\Big| \longrightarrow 0 \text{ as } t'\uparrow t,$$

where the last convergence is a consequence of $\big|F_\eta(t-\tau,\xi)-F_\eta(t'-\tau,\xi)\big||\widehat{uu_x}(\tau,\xi)| \le 2e^{\eta T}|\widehat{uu_x}(\tau,\xi)| \in L^1([0,t],d\tau)$, and the dominated convergence theorem.
On the other hand we have that

$$|(u(t)-u(t'),f)_{X_{\sigma(T)}}| \le \sum_{j=0}^{1} I_1^j(t,t') + I_2(t,t') + I_3(t,t'),$$

where

$$I_1^j(t,t') \equiv \Big|\int h_j(\xi,T)(F_\eta(t,\xi)-F_\eta(t',\xi))\hat{\phi}(\xi)\overline{\hat{f}(\xi)}d\xi\Big|,$$

which, by the dominated convergence theorem, tends to zero as $t'\uparrow t$.
Now we estimate $I_2(t,t')$ defined below. First we see that

$$\begin{aligned}\int|\xi||h_j(\xi,T)|\hat{f}(\xi)|^2d\xi \;&\le\; \cosh(2\sigma(T))\|f\|^2 + \int_{|\xi|>1}|\xi|^{2j+1}\cosh(2\sigma(T)\xi)|\hat{f}(\xi)|^2d\xi \\ &\qquad + \sinh(2\sigma(T))\|f\|^2 + \int_{|\xi|>1}|\xi|^{2j+2}\cosh(2\sigma(T)\xi)|\hat{f}(\xi)|^2d\xi \\ &\le\; e^{2\sigma(T)}\|f\|^2 + 2\|f\|^2_{Y_{\sigma(T)}}. \qquad (38)\end{aligned}$$

Moreover, for every $\tau\in[0,T]$, we have that

$$\|u(\tau)^2\| \le \sup_{x\in\mathbb{R}}|u(\tau,x)|\|u(\tau)\| \le c\|u(\tau)\|_1\|u(\tau)\| \le c\|u(\tau)\|^2_{X_{\sigma(T)}} \qquad (39)$$

and

$$\begin{aligned}\|u(\tau)^2\|_{Y_{\sigma(T)}} &\le \sqrt{2}\|U(\tau)^2\|_{H^{2,2}(\sigma(T))} \le c\|U(\tau)\|^2_{H^{2,2}(\sigma(T))} \\ &\le c(T)\big(\|u(\tau)\|^2 + \|u(\tau)\|^2_{Y_{\sigma(T)}}\big), \qquad (40)\end{aligned}$$

where in the last inequalities we have used Lemmas 4.2 and 4.3, and $U(\tau)$ is the analytic extension of $u(\tau)$ on $S(\sigma(T))$. $I_2(t,t')$ is given by

$$\begin{aligned}I_2(t,t') &\equiv \Big|\sum_{j=0}^{1}\int h_j(\xi,T)\int_{t'}^{t}F_\eta(t-\tau,\xi)\widehat{uu_x}(\tau,\xi)d\tau\overline{\hat{f}(\xi)}d\xi\Big| \\ &= \Big|\sum_{j=0}^{1}\int_{t'}^{t}\int h_j(\xi,T)F_\eta(t-\tau,\xi)\widehat{uu_x}(\tau,\xi)\overline{\hat{f}(\xi)}d\xi d\tau\Big|. \qquad (41)\end{aligned}$$

The interchange of the integrals in (41) is a consequence of Fubini's Theorem since

$$\begin{aligned}
&\int_{t'}^{t}\int h_j(\xi,T)|F_\eta(t-\tau,\xi)||\widehat{uu_x}(\tau,\xi)||\overline{\hat{f}(\xi)}|d\xi d\tau\\
\leq\ & \frac{e^{\eta T}}{2}\int_{t'}^{t}\Big(\int |\xi|h_j(\xi,T)|\hat{f}(\xi)|^2 d\xi\Big)^{1/2}\Big(\int |\xi|h_j(\xi,T)|\widehat{u(\tau)^2}(\xi)|^2 d\xi\Big)^{1/2} d\tau\\
\leq\ & c(T)(\|f\|+\|f\|_{Y_{\sigma(T)}})\int_0^T \big(\|u(\tau)^2\|+\|u(\tau)^2\|_{Y_{\sigma(T)}}\big)d\tau\\
\leq\ & c(T)(\|f\|+\|f\|_{Y_{\sigma(T)}})\int_0^T \big(\|u(\tau)\|^2_{X_{\sigma(T)}}+\|u(\tau)\|^2+\|u(\tau)\|^2_{Y_{\sigma(T)}}\big)d\tau\\
\leq\ & c(T)(\|f\|+\|f\|_{Y_{\sigma(T)}})\big(\|u\|^2_{B(T)}T+\frac{\|u\|^2_{B(T)}}{A}\big)<+\infty, \qquad (42)
\end{aligned}$$

where the second and third last inequalities were consequence of (38)-(40). So, it follows from (41) that

$$I_2(t,t')=\Big|\int_{t'}^{t}\big(u(\tau)\partial_x u(\tau),g(t,\tau)\big)_{X_{\sigma(T)}}d\tau\Big|,$$

where $\overline{\widehat{g(t,\tau)}(\xi)}\equiv F_\eta(t-\tau,\xi)\overline{\hat{f}(\xi)}$. Then $\|g(t,\tau)\|_{X_{\sigma(T)}}\leq e^{\eta T}\|f\|_{X_{\sigma(T)}}$ and the same thing for the $Y_{\sigma(T)}$-norm. By Corollary 4.1 we have that

$$\begin{aligned}
I_2(t,t') &\leq \int_{t'}^{t}\tilde{a_1}(\|u(\tau)\|_{X_{\sigma(T)}})\|u(\tau)\|_{Y_{\sigma(T)}}\big(\|g(t,\tau)\|_{X_{\sigma(T)}}+\|g(t,\tau)\|_{Y_{\sigma(T)}}\big)d\tau\\
&\leq \tilde{a_1}(\|u\|_{B(T)})e^{\eta T}\big(\|f\|_{X_{\sigma(T)}}+\|f\|_{Y_{\sigma(T)}}\big)\Big(\int_{t'}^{t}\|u(\tau)\|^2_{Y_{\sigma(T)}}d\tau\Big)^{1/2}\sqrt{t-t'},
\end{aligned}$$

where $\tilde{a_1}$ is a polynomial with nonnegative coefficients. Last inequality implies that $I_2(t,t')$ tends to zero as $t'\uparrow t$.

Now we estimate $I_3(t,t')$, given by

$$\begin{aligned}
I_3(t,t') &\equiv \Big|\sum_{j=0}^{1}\int h_j(\xi,T)\int_0^{t'}\big(F_\eta(t-\tau,\xi)-F_\eta(t'-\tau.\xi)\big)\widehat{uu_x}(\tau,\xi)d\tau\overline{\hat{f}(\xi)}d\xi\Big|\\
&= \Big|\sum_{j=0}^{1}\int_0^{t'}\int h_j(\xi,T)\big(F_\eta(t-\tau,\xi)-F_\eta(t'-\tau.\xi)\big)\widehat{uu_x}(\tau,\xi)\overline{\hat{f}(\xi)}d\xi d\tau\Big|.
\end{aligned}$$

The interchange of the integrals in the last expression was a consequence of Fubini's Theorem (similar to (41)). So, by the dominated convergence theorem, we have that $I_3(t,t')$ tends to zero as $t'\uparrow t$ since

$$\begin{aligned}
&\big|h_j(\xi,T)\big(F_\eta(t-\tau,\xi)-F_\eta(t'-\tau,\xi)\big)\widehat{uu_x}(\tau,\xi)\overline{\hat{f}(\xi)}\big|\\
\leq\ & e^{\eta T}h_j(\xi,T)|\xi|\big|\widehat{u(\tau)^2}(\xi)\big||\hat{f}(\xi)|\in L^1(\mathbb{R}\times[0,t],d\xi d\tau),
\end{aligned}$$

the last expression is obtained with similar calculations to (42).

Hence $|(u(t)-u(t'),f)_{X_{\sigma(T)}}|\to 0$ as $t'\uparrow t$ (similarly as $t'\downarrow t$). Then $u\in C^\omega([0,T];X_{\sigma(T)})$.

Finally, when $\eta = 0$ we have that $u \in C([0,T]; X_{\sigma(T)})$, as it was already mentioned by Hayashi in Theorem 2 of [6]. In fact in this case, we have that $\frac{1}{2}\frac{d}{dt}\|u(t)\|^2_{X_{\sigma(T)}} = -\Re(u(t)\partial_x u(t), u(t))_{X_{\sigma(T)}}$. Integrating the last expression between t' ant t and using again Corollary 4.1, we obtain

$$\begin{aligned}\big|\|u(t)\|^2_{X_{\sigma(T)}} - \|u(t')\|^2_{X_{\sigma(T)}}\big| &= 2\Big|\int_{t'}^{t} \Re(u(\tau)\partial_x u(\tau), u(\tau))_{X_{\sigma(T)}} d\tau\Big| \\ &\leq 2\tilde{a_1}(\|u\|_{B(T)})\big[\|u\|_{B(T)}\big(\int_{t'}^{t}\|u(\tau)\|^2_{Y_{\sigma(T)}} d\tau\big)^{1/2}\sqrt{t-t'} \\ &\quad + \int_{t'}^{t}\|u(\tau)\|^2_{Y_{\sigma(T)}} d\tau\big] \longrightarrow 0 \text{ as } t' \uparrow t,\end{aligned}$$

and similarly as $t' \downarrow t$. Since $X_{\sigma(T)}$ is a Hilbert space, it follows that $u \in C([0,T]; X_{\sigma(T)})$ when $\eta = 0$. $\square$

Finally, it should be mentioned that Bona, Grujić and Kalisch (see [3]) have recently obtained algebraic lower bounds on the rate of decrease in time of the uniform radius of spatial analyticity for the generalized KdV equation. This raises a potentially interesting question for future research related to the initial value problems considered in this paper.

Appendix

Kato's Inequality: (See [7]).
Let $s > 3/2$, $t \geq 1$. If f and u are real-valued, then

$$|(f\partial_x u, u)_t| \leq C\big(\|\partial_x f\|_{s-1}\|u\|_t^2 + \|\partial_x f\|_{t-1}\|u\|_s\|u\|_t\big). \qquad (K)$$

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